# An Improved Upper Bound for SAT 

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#### Abstract

We show that the CNF satisfiability problem can be solved $O^{*}\left(1.2226^{m}\right)$ time, where $m$ is the number of clauses in the formula, improving the known upper bounds $O^{*}\left(1.234^{m}\right)$ given by Yamamoto 15 years ago and $O^{*}\left(1.239^{m}\right)$ given by Hirsch 22 years ago. By using an amortized technique and careful case analysis, we successfully avoid the bottlenecks in previous algorithms and get the improvement.


## 1 Introduction

The problem of testing the satisfiability of a propositional formula in conjunctive normal form (CNF), denoted by SAT, is one of the most fundamental problems in computer science. It is the first problem proved to be NP-complete [2] and plays an important role in computational complexity and artificial intelligence [9]. To make the problem tractable, a large number of references studied it from the view of heuristic algorithms, approximation algorithms, randomized algorithms, and exact algorithms. In this paper, we study exact algorithms for SAT with guaranteed theoretical running time bounds.

### 1.1 Related Works

To evaluate the running time bound, there are three frequently used measures: the number of variables $n$, the number of clauses $m$, and the length of the whole input $L$. The trivial algorithm to check all possible assignments runs in $O^{*}\left(2^{n}\right)$ timel . A nontrivial bound better than $O^{*}\left(2^{n}\right)$ was obtained in [5], which is $O^{*}\left(2^{n(1-2 \sqrt{1 / n \log m})}\right)$. Later better upper bounds were introduced in [6] and [21]. However, no algorithm with running time bound $O^{*}\left(c^{n}\right)$ for some constant $c<2$ was found, despite decades of hard work. The nonexistence of these algorithms is known as the Strong Exponential Time Hypothesis (SETH) [14. On the other hand, for a restricted version, the $k$-SAT problem (where each clause in the CNF-formula contains at most $k$ literals), a series of significant results have been developed. A branch-and-bound technique was introduced in [17] and [3], which can solve $k$-SAT in $O^{*}\left(\left(\alpha_{k}\right)^{n}\right)$ time where $\alpha_{k}$ is the largest root of the function $x=2-1 / x^{k-1}$. After this, a series of improvements on the upper bounds for $k$-SAT have been made. Most of them are based on derandomization, such as the $O^{*}\left(2^{(1-1 / 2 k) n}\right)$ bound in [19]

[^0]and the $O^{*}\left((2-2 /(k+1))^{n}\right)$ bound in [4]. Recently a new randomized algorithm for $k$-SAT with better running time bound was introduced [11].

When the length of the input $L$ is taken as the measure, from the first algorithm with running time bound $O^{*}\left(1.0927^{L}\right)$ by Gelder [10], the result was improved frequently. Let us quote the bound $O^{*}\left(1.0801^{L}\right)$ by Kullmann [15], $O^{*}\left(1.0758^{L}\right)$ by Hirsch [12], $O^{*}\left(1.074^{L}\right)$ by Hirsch [13], and $O^{*}\left(1.0663^{L}\right)$ by Wahlström [22]. Currently, the best known bound was $O^{*}\left(1.0652^{L}\right)$ obtained by Chen and Liu [1].

Another important measure is the number of clauses $m$. Monien and Speckenmeyer [16] gave an $O^{*}\left(1.260^{m}\right)$-time algorithm in 1980, which was improved to $O^{*}\left(1.239^{m}\right)$ by Hirsch [12] in 1998. Then it took seven years for Yamamoto to slightly improve Hirsch's bound to $O^{*}\left(1.234^{m}\right)$ [23]. In this paper, we will significantly improve Yamamoto's bound obtained 15 years ago. Previous and our results are listed in Table 1

Table 1: Previous and our upper bounds for SAT

| running times | references |
| :--- | :---: |
| $O^{*}\left(1.260^{m}\right)$ | $[16$ |
| $O^{*}\left(1.239^{m}\right)$ | $[12$ |
| $O^{*}\left(1.234^{m}\right)$ | $[23]$ |
| $O^{*}\left(1.2226^{m}\right)$ | This paper |

### 1.2 The Techniques

All algorithms in Table 1 are branch-and-search algorithms. The branch-and-search idea is simple and practical: we iteratively branch on a literal into two branches by letting it be 1 or 0 . Consider an $(a, b)$-literal (a literal such that itself appears in $a$ clauses and the negation of it appears in $b$ clauses). In the branching where the literal is assigned 1 , we can reduce $a$ clauses; in the branching where the literal is assigned 0 , we can reduce $b$ clauses. We hope that the values of $a$ and $b$ are larger, so that we can reduce the instance to a greater extent. There are several developed techniques to deal with $(a, b)$-literals with small values of $a$ and $b$, say one of them is at most 2 . Thus the worst case will become to branch on a (3,3)-literal, in which we can only get a branching vector of $(3,3)$ and a branching factor 1.2600 . We get the bound of $O^{*}\left(1.260^{m}\right)$ [16]. It seems that branching on (3,3)-literals is unavoidable. Hirsch [12] showed that after branching on a (3,3)-literal we can always branch with a branching vector at least $(4,3)$ or $(3,4)$ subsequently. Combing the bad branching vector $(3,3)$ with the good branching vector $(4,3)$ or $(3,4)$, he got a better worst-case and then improved the running time bound to $O^{*}\left(1.239^{m}\right)$. Yamamoto [23] further showed that the worst cases in Hirsch's algorithm would not always happen: we can further branch with $(4,3)$ or $(3,4)$ at the third level, i.e., after branching with $(4,3)$ or $(3,4)$ after branching with $(3,3)$. Yamamoto considered more levels of the branching but could only slightly improve the bound to $O^{*}\left(1.234^{m}\right)$. The improvement is very slow, and we seem to have reached the bottleneck.

Our algorithm is still a branch-and-search algorithm, following the main framework in the previous algorithms. We still can not avoid branching on (3,3)-literals, otherwise the worst case would be to branch on (3,4)-literals or (4,3)-literals and the bound would be improved to $O^{*}\left(1.2208^{m}\right)$. We also show that after branching on a (3,3)-literal we can further branch with better branching vectors. However, the traditional analysis to combine several levels of branchings into a big branching is somewhat complicated and limited. To exhibit the relations among good and bad branchings in our algorithm and also to use as many good branchings as possible to even out the bad ones, we will use an amortized technique to analyze the running time bound. To get the claimed result, we also need to use some new reduction and branching
rules and deep analysis of the structure.

## 2 Preliminaries

Let $V=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ denote a set of $n$ boolean variables. For each variable $x_{i}(i=$ $1,2,3, \ldots, n$ ), a literal is either $x_{i}$ or the negation of it $\overline{x_{i}}$ (we use $\bar{x}$ to denote the negation of a literal $x$, and then $\overline{\bar{x}}=x$ ). A clause on $V$ is a set of literals on $V$ without a negation of any literal in it, which means $x$ and $\bar{x}$ cannot be contained simultaneously in a clause for any variable $x \in V$. A $C N F$-formula on $V$ is a sequence of clauses $\mathcal{F}=\left\{C_{1}, C_{2}, C_{3}, \ldots, C_{m}\right\}$. We will use $m_{\mathcal{F}}$ to denote the number of clauses in $\mathcal{F}$. An assignment for $V$ is a map $A: V \rightarrow\{0,1\}$. A clause $C_{j}$ on $V$ is satisfied by $A$ if and only if there exists a literal $x$ in $C_{j}$ such that $A(x)=1$. A CNF-formula is satisfied by an assignment $A$ if and only if each clause in it is satisfied by $A$. An assignment $A$ that makes a CNF-formula $\mathcal{F}$ satisfied is called a satisfying assignment for $\mathcal{F}$. Given a CNF-formula $\mathcal{F}$ on a set of variables $V$, the SAT problem is to check the existence of a satisfying assignment for $\mathcal{F}$.

The degree of a literal $x$ in $\mathcal{F}$ is the number of clauses in $\mathcal{F}$ containing it. The total degree of a literal $x$ is the degree of $x$ plus the degree of $\bar{x}$. If the degree of $x$ is $a$ (resp., at least $a$ or at most $a$ ) and the degree of $\bar{x}$ is $b$, we say $x$ is an ( $a, b$ )-literal (resp., an ( $a^{+}, b$ )-literal or an $\left(a^{-}, b\right)$-literal $)$. Similarly, we can define $\left(a, b^{+}\right)$-literal, ( $\left.a, b^{-}\right)$-literal, $\left(a^{+}, b^{+}\right)$-literal, $\left(a^{-}, b^{-}\right)$literal and so on. Note that a literal $x$ is an $(a, b)$-literal if and only if $\bar{x}$ is a $(b, a)$-literal. A clause containing exactly $c$ literals is called a $c$-clause. A pair of literals $x$ and $y$ is called a coincident pair if there are at least two clauses containing them simultaneously.

Our algorithm will first apply reduction rules to reduce the instance and then apply branching rules to search for a solution when the instance can not be further reduced. Next, we first introduce the reduction rules.

## 3 Reduction Rules

We have five reduction rules. The first two are easy to observe and used in the literature [7].
R-Rule 1 (Elimination of 1-clauses and pure literals) If the CNF-formula contains a 1 -clause $\{x\}$ or an ( $a, 0$ )-literal $x$ with $a>0$, assign $x=1$.

R-Rule 2 (Elimination of subsumptions) If the CNF-formula contains two clauses $C$ and $C^{\prime}$ such that $C \subseteq C^{\prime}$, then delete $C^{\prime}$.

The following proposition is known as the resolution technique in the literature, which was first proved in [20], and then used in many SAT algorithms.

Definition 1 (Resolution on a variable) Let $\mathcal{F}$ be a CNF-formula containing a variable $x$. Let $E_{1}, E_{2}, \ldots, E_{a}$ be the clauses containing $x$ and $D_{1}, D_{2}, \ldots, D_{b}$ be the clauses containing $\bar{x}$. Resolving on variable $x$ is to construct a new CNF-formula $\mathcal{F}_{\backslash x}$ by the following method: for each $i \in\{1,2, \ldots, a\}$ and $j \in\{1,2, \ldots, b\}$, add the clause $F_{i j}=E_{i} \cup D_{j} \backslash\{x, \bar{x}\}$ to the formula if it does not contain both a literal and the negation of it; delete $E_{i}(i \in\{1,2, \ldots, a\})$ and $D_{j}$ ( $j \in\{1,2, \ldots, b\}$ ) from the formula.

We may always use $\mathcal{F}_{\backslash x}$ to denote the CNF-formula after resolving a variable $x$ in $\mathcal{F}$.
Proposition 1 [20] Let $\mathcal{F}$ be a CNF-formula containing a variable $x$ and $\mathcal{F}_{\backslash x}$ be the CNFformula after resolving on variable $x$. Then $\mathcal{F}$ has a satisfying assignment if and only if $\mathcal{F}_{\backslash x}$ does.

R-Rule 3 (Resolving on some variables) If there is an ( $a, b$ )-literal $x$ such that $a=1$ and $b \geq 1$ or $a=2$ and $b=2$, then resolve $x$ in $\mathcal{F}$, i.e., replace $\mathcal{F}$ with $\mathcal{F}_{\backslash x}$.

We also introduce a simple but powerful concept, based on which we can design several reduction rules.

Definition 2 (Autarkic sets) $A$ set $X$ of literals is called an autarkic set if each clause containing a negation of a literal in $X$ also contains a literal in $X$.

Lemma 1 If a CNF-formula $\mathcal{F}$ has a satisfying assignment, then it has a satisfying assignment where all literals in an autarkic set are assigned 1.

Proof. If we assign 1 to all literals in an autarkic set $X$, then any clause containing either a literal in $X$ or a negation of a literal in $X$ is satisfied, since each clause containing a negation of a literal in $X$ also contains a literal in $X$. Any other assignment of literals in $X$ can only satisfy a subset of these clauses. So we can simply assign 1 to all literals in $X$.

The following reduction rule was firstly used in [12]. It is an application of a special autarkic set.

R-Rule 4 [12] If each clause containing a $\left(2,3^{+}\right)$-literal also contains a $\left(3^{+}, 2\right)$-literal, assign 1 to each $\left(3^{+}, 2\right)$-literal.

Our algorithm also needs to eliminate another kind of autarkic sets.
R-Rule 5 Let $X$ be the set of $(4,3)$-literals $x$ such that there is a clause containing both $x$ and $a$ $\left(3,3^{+}\right)$-literal. If each clause containing a negation of a literal in $X$ also contains a $(4,3)$-literal, assign 1 to each literal in $X$.

Each clause containing a negation of a literal $x \in X$ also contains a (4,3)-literal $y$. Since $\bar{x}$ is a $(3,4)$-literal, we know that $y$ is also in $X$. Thus $X$ is an autarkic set. In this reduction rule, the requirement of 'a clause containing both $x$ and a $\left(3,3^{+}\right)$-literal' plays no role in establishing $X$ to be an autarkic set. This requirement is used to identify a particular subset of $(4,3)$-literals, which will be useful in our analysis.

Lemma 2 After applying any of the above reduction rules, the satisfiability of the formula does not change. Except for the application of $R$-Rule 3 on a (2,2)-literal where the number of clauses does not increase, each application of other reduction rules decreases the clause number by at least 1.

Definition 3 (Reduced formulas) A formula is called reduced if none of the five reduction rules can be applied on the formula.

For an instance $\mathcal{F}$, we will use $R(\mathcal{F})$ to denote the resulting reduced formula after iteratively applying the reduction rules on $\mathcal{F}$.

Lemma 3 Given a formula, we can apply the five reduction rules in polynomial time to change it to a reduced formula.

Proof. It is easy to see that each reduction rule can be applied in polynomial time. Since each reduction rule either assigns a literal to 1 or resolve a variable, we know that we can apply at most $n$ times of reduction rules. Thus, the total running time is bounded by a polynomial.

Lemma 4 Let $\mathcal{F}$ be a reduced formula. Then there is no 1-clause, (2,2)-literal or ( $\left.1^{-}, a\right)$-literal with $a \geq 1$ in $\mathcal{F}$. Furthermore, the total degree of any literal in $\mathcal{F}$ is at least 5 .

Proof. If there is a $(0, a)$-literal, then R-Rule 1 would be applicable. If there is a $(1, a)$-literal, then R-Rule 3 would be applicable. If there is a ( 2,2 )-literal, then R-Rule 3 would be applicable. If there is a 1 -clause, then R -Rule 1 would be applicable. All these contradict the fact that $\mathcal{F}$ is reduced.

If a literal has a total degree at most 4 , then it must be a $(2,2)$ or $\left(1^{-}, a\right)$ or $\left(a, 1^{-}\right)$-literal. For the last case, the negation of the literal is a $\left(1^{-}, a\right)$-literal.

## 4 Branch-and-Search Paradigms

Our algorithm will first apply our reduction rules to reduce the instance. When no reduction rule can be applied anymore, we will branch to search for a solution. Our branching rule is simple. We take a literal $x$ and branch on it into two sub-instances. In one sub-instance we assign $x=1$ and in the other one we assign $x=0$, i.e, we get two sub instances $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$. Selecting different literals to branch will lead to different algorithms. We want to select 'good' literals to branch on such that the size of the sub instances can be reduced fast.

We use the number $m$ of clauses to evaluate the size of the formula. Assume the number of clauses of the current instance is $m$. If a branching operation branches into $l$ sub-branches such that the number of clauses in the $i$-th sub-instance decreases by at least $c_{i}$, we say this operation branches with a branching vector $\left(c_{1}, c_{2}, \ldots, c_{l}\right)$. The largest root of the function $f(x)=1-\sum_{i=1}^{l} x^{-c_{i}}$ is called the branching factor. If $\gamma$ is the maximum branching factor among all branching factors in an algorithm, then the running time of the algorithm is bounded by $O^{*}\left(\gamma^{m}\right)$. More details about the analysis and how to solve recurrences can be found in the monograph [8]. The following property is frequently used in the paper: for two branching vectors $C=\left(c_{1}, c_{2}, \ldots, c_{l}\right)$ and $B=\left(b_{1}, b_{2}, \ldots, b_{l}\right)$, if it holds that $c_{i} \geq b_{i}$ for each $i$, then we say $B$ covers $C$. The corresponding branching factor of a branching vector $C$ is not greater than the corresponding branching factor of a branching vector that covers $C$.

### 4.1 Good formulas \& bad formulas

Similar to the technique used by Niedermeier and Rossmanith to solve the 3-hitting set problem [18], we also classify formulas in our algorithm into two classes: good formulas and bad formulas. For good formulas, we may be able to branch with good branching vectors. For bad formulas, we may only be able to get bad branching vectors. We will show that bad formulas will not appear frequently. Then we can use an amortized analysis to get better branching vectors. To make the amortized analysis easy to follow, we will use the substitution method to prove our bounds. The precise definitions of good and bad formulas are given below.

Definition 4 (Good formulas \& bad formulas) A formula $\mathcal{F}$ is $a$ bad formula if and only if the following four conditions are satisfied
(1) $\mathcal{F}$ only contains $(3,3)$-literals, $(3,4)$-literals and $(4,3)$-literals.
(2) There is no coincident pair.
(3) There is no 2-clause.
(4) There is no clause containing a $(4,3)$-literal and a $\left(3,3^{+}\right)$-literal simultaneously.

A formula is good if it is not a bad formula.

### 4.2 The algorithm and its analysis

The main steps of our algorithm are listed in Algorithm 1. The precise descriptions and analysis of lines 11 and 14 are delayed to Sections 6.1 and 6.2 .

```
Algorithm \(1 \operatorname{SAT}(\mathcal{F})\)
    if \(\{\mathcal{F}\) is not reduced \(\}\) then
        Iteratively apply our reduction rules to reduce it.
    end if
    if \(\{\mathcal{F}\) is empty \(\}\) then
        Return true.
    end if
    if \(\{\mathcal{F}\) contains an empty clause \(\}\) then
        Return false.
    end if
    if \(\{\mathcal{F}\) is a bad formula \(\}\) then
        Apply branching rules in Sec. 6.1 to search for a solution.
    end if
    if \(\{\mathcal{F}\) is a good formula \(\}\) then
        Apply branching rules in Sec. 6.2 to search for a solution.
    end if
```

Recall that, for an instance $\mathcal{F}, R(\mathcal{F})$ is the resulting reduced instance after applying the reduction rules on $\mathcal{F}$, and $m_{\mathcal{F}}$ is the number of clauses in $\mathcal{F}$. We have the following important lemmas, which are the base for us to establish the running time bound.

Lemma 5 Let $\mathcal{F}$ be a CNF-formula. It holds that $m_{R(\mathcal{F})} \leq m_{\mathcal{F}}$. Furthermore, if $\mathcal{F}$ is good, then either $R(\mathcal{F})$ is good or $m_{R(\mathcal{F})} \leq m_{\mathcal{F}}-1$.

Proof. By Lemma 2, we have that $m_{R(\mathcal{F})} \leq m_{\mathcal{F}}$. Next, we assume that $\mathcal{F}$ is good.
If $R(\mathcal{F})=\mathcal{F}$, obviously $R(\mathcal{F})$ is good. So we assume that some R-Rules are applied. By Lemma 2, we know that if $m_{R(\mathcal{F})}=m_{\mathcal{F}}$ then only R-Rule 3 is applied on $(2,2)$-literals. For any $\mathcal{F}^{\prime}$ with a (2,2)-literal $x$ in it, we show that after applying R-Rule 3 on $x$ the resulting instance $\mathcal{F}_{{ }_{x}}^{\prime}$ is good. Let the two clauses containing $x$ in $\mathcal{F}^{\prime}$ be $D_{1}$ and $D_{2}$, the two clauses containing $\bar{x}$ be $E_{1}$ and $E_{2}$. If $m_{\mathcal{F}^{\prime}}=m_{\mathcal{F}_{\backslash x}^{\prime}}$, then all $E_{i j}=D_{i} \cup E_{j} \backslash\{x, \bar{x}\}$ for each $1 \leq i, j \leq 2$ are in $\mathcal{F}_{\backslash x}^{\prime}$. If one of $D_{1}, D_{2}, E_{1}$ and $E_{2}$ contains at least three literals, then we will get some coincident pair. Otherwise, each $E_{i j}$ is a 2-clause. For any case, $\mathcal{F}_{\backslash x}^{\prime}$ is good.

Lemma 6 If the formula $\mathcal{F}$ is reduced and bad, then our algorithm can branch with either a branching vector covered by $(3,4)$ or $(4,3)$, or a branching vector $(3,3)$ such that the formula in each branch is good.

Lemma 7 If the formula to branch is reduced and good, then our algorithm can branch with either a branching vector covered by one of $(3,5),(5,3)$, and $(4,4)$, or a branching vector $(3,4)$ or $(4,3)$ such that the formula in each branch is good.

The proof of Lemma 6 and Lemma 7 are given in Sections 6.1 and 6.2 respectively. Next, we prove the running time bound of the algorithm based on Lemma 5, Lemma 6, and Lemma 7 ,

Theorem 1 SAT can be solved in $O^{*}\left(1.2226^{m}\right)$ time.

Proof. We use $T(\mathcal{F})$ to denote the size of the search tree generated by the algorithm running on an instance $\mathcal{F}$. We only need to prove that $T(\mathcal{F})=O\left(1.2226^{m_{\mathcal{F}}}\right)$. To prove the theorem, we will show that there are two constants $c_{1}=2$ and $c_{2}=c_{1} / 0.9136$ such that

$$
\begin{equation*}
T(\mathcal{F}) \leq c_{1} 1.2226^{m_{\mathcal{F}}}-1, \quad \text { if } \mathcal{F} \text { is good, } \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
T(\mathcal{F}) \leq c_{2} 1.2226^{m_{\mathcal{F}}}-1, \quad \text { if } \mathcal{F} \text { is bad. } \tag{2}
\end{equation*}
$$

First of all, we show that we can assume $\mathcal{F}$ is a reduced instance without loss of generality. If the current instance $\mathcal{F}$ with $m$ clauses is not a reduced one, our algorithm will apply reduction rules on it to get a reduced instance $\mathcal{F}^{*}$ with $m^{*}$ clauses. To prove that (1) and (2) hold for $\mathcal{F}$, we only need to prove that (1) and (2) hold for $\mathcal{F}^{*}$. The reason is based on the following observations. If both of $\mathcal{F}$ and $\mathcal{F}^{*}$ are bad or good, then it holds that $c_{i} 1.2226^{m_{\mathcal{F}^{*}}} \leq c_{i} 1.2226^{m_{\mathcal{F}}}$ since $m_{\mathcal{F}^{*}} \leq m_{\mathcal{F}}$ by Lemma 5 , If $\mathcal{F}$ is bad and $\mathcal{F}^{*}$ is good, then it holds that $c_{1} 1.2226^{m_{\mathcal{F}^{*}} \leq}$ $c_{2} 1.2226^{m_{\mathcal{F}}}$. If $\mathcal{F}$ is good and $\mathcal{F}^{*}$ is bad, then it still holds that $c_{2} 1.2226^{m_{\mathcal{F}^{*}}} \leq c_{1} 1.2226^{m_{\mathcal{F}}}$ because now we have $m_{\mathcal{F}^{*}} \leq m_{\mathcal{F}}-1$ by Lemma 5 and then $c_{1}<1.2226 c_{2}$.

Next, we simply assume that the instance $\mathcal{F}$ is reduced and use $\mathcal{F}_{1}$ and $\mathcal{F}_{2}$ to denote the two sub instances generated by our branching operations. We use the substitution method to prove (1) and (2).

Assume that $T(\mathcal{F}) \leq c_{i} 1.2226^{m_{\mathcal{F}}}-1$ (where $c_{i}=c_{1}$ if $\mathcal{F}$ is good and $c_{i}=c_{2}$ if $\mathcal{F}$ is bad) holds for all instances $\mathcal{F}$ with less than $m$ clauses. We show that it also holds for instances with $m$ clauses.

First, we consider the case where $\mathcal{F}$ is bad. According to Lemma 6, there are two cases. For the first case of branching with a vector $(3,4)$ or $(4,3)$, we have that

$$
\begin{aligned}
T(\mathcal{F})= & T\left(R\left(\mathcal{F}_{1}\right)\right)+T\left(R\left(\mathcal{F}_{2}\right)\right)+1 \\
& \leq c_{2} 1.2226^{m_{R\left(\mathcal{F}_{1}\right)}}+c_{2} 1.2226^{m_{R\left(\mathcal{F}_{2}\right)}}-1 \\
& \left.\quad \text { (by the assumption and } c_{1}<c_{2}\right) \\
& \leq c_{2} 1.2226^{m_{\mathcal{F}}-3}+c_{2} 1.2226^{m_{\mathcal{F}}-4}-1 \\
\leq & c_{2} 1.2226^{m_{\mathcal{F}}}-1 .
\end{aligned}
$$

For the second case of branching with a vector (3,3), the two sub instances are good, we have that

$$
\begin{aligned}
T(\mathcal{F}) & =T\left(R\left(\mathcal{F}_{1}\right)\right)+T\left(R\left(\mathcal{F}_{2}\right)\right)+1 \\
& \leq c_{1} 1.2226^{m_{\mathcal{F}}-3}+c_{1} 1.2226^{m_{\mathcal{F}}-3}-1 \\
& \leq c_{2} 1.2226^{m_{\mathcal{F}}}-1 .
\end{aligned}
$$

Second, we consider the case where $\mathcal{F}$ is good. According to Lemma 7 there are two cases.
In the first case, the branching vector is $(3,5)$ or $(5,3)$ or $(4,4)$. If it is $(3,5)$ or $(5,3)$, we have that

$$
\begin{aligned}
T(\mathcal{F}) & =T\left(R\left(\mathcal{F}_{1}\right)\right)+T\left(R\left(\mathcal{F}_{2}\right)\right)+1 \\
& \leq c_{i_{1}} 1.2226^{m_{\mathcal{F}}-3}+c_{i_{2}} 1.2226^{m_{\mathcal{F}}-5}-1 \\
& \leq c_{2} 1.2226^{m_{\mathcal{F}}-3}+c_{2} 1.2226^{m_{\mathcal{F}}-5}-1 \\
& \leq c_{1} 1.2226^{m_{\mathcal{F}}}-1,
\end{aligned}
$$

where $c_{i_{1}}, c_{i_{2}} \in\{1,2\}$. If the branching vector is $(4,4)$, we have that

$$
\begin{aligned}
T(\mathcal{F}) & =T\left(R\left(\mathcal{F}_{1}\right)\right)+T\left(R\left(\mathcal{F}_{2}\right)\right)+1 \\
& \leq c_{i_{1}} 1.2226^{m_{\mathcal{F}}-4}+c_{i_{2}} 1.2226^{m_{\mathcal{F}}-4}-1 \\
& \leq 2 c_{2} 1.2226^{m_{\mathcal{F}}-4}-1 \\
& \leq c_{1} 1.2226^{m_{\mathcal{F}}}-1
\end{aligned}
$$

where $c_{i_{1}}, c_{i_{2}} \in\{1,2\}$.
For the second case of branching with a vector $(3,4)$ or $(4,3)$ such that the two sub instances are good, we have that

$$
\begin{aligned}
T(\mathcal{F}) & =T\left(R\left(\mathcal{F}_{1}\right)\right)+T\left(R\left(\mathcal{F}_{2}\right)\right)+1 \\
& \leq c_{1} 1.2226^{m_{\mathcal{F}}-3}+c_{1} 1.2226^{m_{\mathcal{F}}-4}-1 \\
& \leq c_{1} 1.2226^{m_{\mathcal{F}}}-1
\end{aligned}
$$

We have proved that (11) and (21) hold for $\mathcal{F}$. Thus, it holds that $T(\mathcal{F})=O\left(1.2226^{m_{\mathcal{F}}}\right)$, no matter $\mathcal{F}$ is good or bad.

## 5 Some Properties

Before giving the detailed steps of the branching operations, we give some properties that will be used to simplify our presentation and analysis.

In a branching operation, we need to analyze the branching vector, i.e., the number of clauses decreased in each branching. Sometimes we can get a branching vector good enough for our analysis, such as branching vectors $(4,4),(3,5)$ and $(5,3)$. Sometimes the branching vector is not good enough and we still need to prove the remaining formulas are good, which will allow us to use amortization. Usually, we will fall in one of the following two cases:

1. Some variables are assigned values (including applying R-Rule 1) and then some clauses are deleted because some literals in them are assigned 1 . We need to prove that the remaining formula is good.
2. R-Rule 3 is applied and we need to prove that the remaining formula is good.

We will use the following two lemmas to help us solve these two cases.
Lemma 8 Let $\mathcal{F}$ be a formula containing a $\left(3^{-}, 0^{+}\right)$or $\left(0^{+}, 2^{-}\right)$-literal $y$. Assume the total degree of $y$ is $a>0$. If we delete from $\mathcal{F}$ at most $a-1$ clauses and some literals other than $y$ and $\bar{y}$, where at least one deleted clause contains $y$, then the resulting formula is good.

Proof. Since the total degree of $y$ is $a$, at least one clause containing $y$ or $\bar{y}$ will not be deleted. Then $y$ or $\bar{y}$ will be a $\left(2^{-}, 0^{+}\right)$-literal in the remaining formula. Thus the formula is good.

Corollary 1 Let $\mathcal{F}$ be a reduced formula containing only $\left(3^{-}, 3^{-}\right),\left(2,4^{+}\right)$and $\left(4^{+}, 2\right)$-literals. For any literal $x$ in it with degree at most 4 , the formula $\mathcal{F}_{x}$ is good.

Proof. By Lemman we know that the total degree of any literal in $\mathcal{F}$ is at least 5 and $\mathcal{F}$ does not contain any 1 -clauses. Note that $\mathcal{F}_{x}$ is obtained from $\mathcal{F}$ by deleting all clauses containing $x$ and deleting the literal $\bar{x}$. Any literal different from $x$ in a clause containing $x$ will be the literal $y$ in Lemma 8, By Lemma 8, we know the corollary holds.

Lemma 9 Let $\mathcal{F}$ be a formula containing a $\left(1,1^{+}\right)$-literal $x$ and at least two different $\left(2^{-}, 0^{+}\right)$literals other than $x$ and $\bar{x}$. It holds that either $m_{\mathcal{F}_{\backslash x}} \leq m_{\mathcal{F}}-1$ and $\mathcal{F}_{\backslash x}$ is a good formula or $m_{\mathcal{F}_{\backslash x}} \leq m_{\mathcal{F}}-2$.

Proof. Let the unique clause containing $x$ be $C$ and the clauses containing $\bar{x}$ be $D_{1}, D_{2}, \ldots D_{l}$. Let $y$ and $z$ be two different $\left(2^{-}, 0^{+}\right)$-literals other than $x$ and $\bar{x}$, where $y$ and $z$ can be each other's negation.

It is easy to see that resolving on $x$ will decrease the number of clauses by at least 1 . We assume that the number of clauses decreases by exactly 1 after resolving on $x$ and show for this case the formula $\mathcal{F}_{\backslash x}$ must be good. For this case, the $l+1$ clauses $C, D_{1}, D_{2}, \ldots D_{l}$ are deleted and all the $l$ clauses $D_{i} \cup C \backslash\{x, \bar{x}\}(i=1,2, \ldots, l)$ are added in $\mathcal{F}_{\backslash x}$.

Case 1. $x$ is a ( 1,1 )-literal: after resolving on $x$, the degree of any literal does not increase and no literal other than $x$ and $\bar{x}$ disappears. So $y$ and $z$ are still $\left(2^{-}, 0^{+}\right)$-literals, witnessing the goodness of $\mathcal{F}_{\backslash x}$.

Case 2. $x$ is a $\left(1,2^{+}\right)$-literal: We further distinguish two cases: $|C| \geq 3$ and $|C| \leq 2$. If $|C| \geq 3$, then any pair of literals in $C \backslash\{x\}$ will be a coincident pair in $\mathcal{F}_{\backslash x}$. Thus, $\mathcal{F}_{\backslash x}$ is good. If $|C| \leq 2$, then at most one literal the degree of who will increase after resolving on $x$, since only the degree of literals in $C \backslash\{x\}$ will increase. So one of $y$ and $z$ will be remained as a $\left(2^{-}, 0^{+}\right)$-literal in $\mathcal{F}_{\backslash x}$. Thus, $\mathcal{F}_{\backslash x}$ is good.

## 6 Detailed branching operations

In this section, we show the detailed branching operations in Algorithm (1) Recall that we only branch on reduced formulas. The detailed branching steps for bad and good formulas are given in Sec. 6.1 and 6.2, respectively. For a bad formula, if there exist $(3,4)$ or $(4,3)$-literals, then deal with them. Else we deal with $(3,3)$-literals. For a good formula, we first deal with $\left(3,5^{+}\right)$or $\left(4^{+}, 4^{+}\right)$-literals; second deal with ( 3,4 )-literals (and also ( 4,3 )-literals); third deal with $\left(2,3^{+}\right)$-literals (and also $\left(3^{+}, 2\right)$-literals); last there are only $(3,3)$-literals and we deal with them.

The main results of these steps are summarized in the following two tables, where the number with ${ }^{\text {'* }}$ ' in the 'Vectors' column means the corresponding branch will leave a good formula. From the two tables, we can see that direct analysis will get a bound of $O^{*}\left(1.2600^{m}\right)$ since the largest branching factor is 1.2600 . This does not use amortization. Our deep analysis in the proof of Theorem 1 shows that we can improve the bound to $O^{*}\left(1.2226^{m}\right)$.

Table 2: Branching for Bad Formulas

| Cases | Literals | Vectors | Factors |
| :--- | :---: | :---: | :---: |
| Case 1 | $(3,4)$-literals | $(3,4)$ | 1.2208 |
| Case 2 | $(3,3)$-literals | $\left(3^{*}, 3^{*}\right)$ | 1.2600 |

## 6.1 $\mathcal{F}$ is a bad formula

Case 1. $\mathcal{F}$ contains a $(3,4)$-literal $x$ : We branch on $x$ into two branchings $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$. The branching vector is $(3,4)$.

Case 2. $\mathcal{F}$ only contains (3, 3)-literals: We branch on an arbitrary literal $x$ into two branchings $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$. The branching vector is $(3,3)$. However, the two sub-instances in the two branchings are good formulas by Corollary 1 .

Table 3: Branching for Good Formulas

| Cases | Literals | Vectors | Factors |
| :--- | :---: | :---: | :---: |
| Case 1 | $\left(3,5^{+}\right)$-literals | $(3,5)$ | 1.1939 |
| Case 1 | $\left(4^{+}, 4^{+}\right)$-literals | $(4,4)$ | 1.1893 |
| Case 2 | $(3,4)$-literals | $(4,4)$ | 1.1893 |
|  |  | $(3,5)$ or $(5,3)$ | 1.1939 |
|  |  | $\left(3^{*}, 4^{*}\right)$ or $\left(4^{*}, 3^{*}\right)$ | 1.2208 |
| Case 3 | $\left(2,3^{+}\right)$-literals | $(4,4)$ | 1.1893 |
|  |  | $(3,5)$ or $(5,3)$ | 1.1939 |
|  |  | $\left(3^{*}, 4^{*}\right)$ or $\left(4^{*}, 3^{*}\right)$ | 1.2208 |
| Case 4 | $(3,3)$-literals | $(4,4)$ | 1.1893 |
|  |  | $(3,5)$ or $(5,3)$ | 1.1939 |
|  |  | $\left(3^{*}, 4^{*}\right)$ or $\left(4^{*}, 3^{*}\right)$ | 1.2208 |

## 6.2 $\mathcal{F}$ is a good formula

Case 1. $\mathcal{F}$ contains a $\left(3,5^{+}\right)$or $\left(4^{+}, 4^{+}\right)$-literal $x$ : Branch on $x$ into two branchings $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$. The branching vector will be at least $(3,5)$ or $(4,4)$.

Case 2. $\mathcal{F}$ contains a (3, 4)-literal (but no $\left(3,5^{+}\right)$or $\left(4^{+}, 4^{+}\right)$-literal): We further distinguish several cases to analyze the branching vector.

Case 2.1. $\mathcal{F}$ also contains a $\left(2,3^{+}\right)$-literal $y$ : We first branch on an arbitrary $(3,4)$-literal $x$ into two branchings $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$. If there is a clause containing both $x$ and $y$, then in the branching $\mathcal{F}_{x}$, the degree of $y$ is at most 1 . Thus $y$ will become a $\left(1,1^{+}\right)$-literal or $\left(0,1^{+}\right)$-literal in $\mathcal{F}_{x}$ and we will further apply R-Rule 1 or 3 on $y$ to decrease the number of clauses by at least 1. We can get a branching vector at least $(4,4)$.

If there is a clause containing both $\bar{x}$ and $y$, then in the branching $\mathcal{F}_{\bar{x}}$, the degree of $y$ is at most 1. We apply R-Rule 1 or 3 on $y$ to further decrease the number of clauses by at least 1 . We can get a branching vector at least $(3,5)$.

The remaining case is that the clauses containing $x$ or $\bar{x}$ does not contain $y$. For this case, we can only get a branching vector (3,4). However, in each branching of $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$, the new instance is a good formula, because there is at least one $\left(2,0^{+}\right)$-literal $y$ in them.

Case 2.2. $\mathcal{F}$ contains only (3,4)-literals, $(4,3)$-literals and $(3,3)$-literals: Let $Y$ be the set of $(4,3)$-literals $x^{\prime}$ such that there is a clause containing both $x^{\prime}$ and a $\left(3,3^{+}\right)$-literal.

Case 2.2.1. $Y \neq \emptyset$ : There is a literal $x \in Y$ and a clause containing $\bar{x}$ which does not contain any $(4,3)$-literals, otherwise R-Rule 5 could be applied and $\mathcal{F}$ would not be a reduced instance. Thus the clause containing $\bar{x}$ will contain some $\left(3,3^{+}\right)$-literals. We branch on $x$ with a branching vector $(4,3)$. By Lemma 8 , we know that both branchings $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$ are good formulas.

Case 2.2.2. $Y=\emptyset$ : For this case, $(4,3)$-literals appear in clauses containing only $(4,3)$ literals. Now Conditions (1) and (4) in the definition of bad formulas hold. Since $\mathcal{F}$ is a good formula now, we know either Condition (2) or Condition (3) will not hold. Thus there is either a 2-clause or a coincident pair.

First, we assume that $\mathcal{F}$ contains a coincident pair $\{x, y\}$. If $x$ is a (3,4)-literal, then $y$ must be a $\left(3,3^{+}\right)$-literal. For this case, we branch on $x$ into two branchings $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$. In the branching $\mathcal{F}_{x}$, literal $y$ becomes a $\left(1,1^{+}\right)$-literal or a $\left(0,1^{+}\right)$-literal and we can reduce the number of clauses by 1 by applying R-Rule 3 or R-Rule 1 on $y$. We get a branching vector $(4,4)$. If both of $x$ and $y$ are $(3,3)$-literals, we branch on an arbitrary $(3,4)$-literal with a branching vector $(3,4)$. Furthermore, in each branching, the instance is a good formula because there is
either a coincident pair $(x, y)$ or one of $x$ and $y$ becomes a literal of degree at most 2 . The remaining case is that both of $x$ and $y$ are (4,3)-literals. For this case, we branch on $x$ into two branchings $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$ with a branching vector (4,3). The formula $\mathcal{F}_{x}$ is good because literal $y$ becomes a $\left(2^{-}, 1^{+}\right)$-literal. The formula $\mathcal{F}_{\bar{x}}$ is good by Lemma $⿴$. Notice that for this case in $\mathcal{F}$ the clauses containing $\bar{x}$ cannot contain any (4,3)-literal and then each of them must contain another ( $3,3^{+}$)-literal.

Second, we assume that $\mathcal{F}$ does not contain any coincident pair and there is a 2 -clause $\{x, y\}$. We branch on $x$ into two branchings $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$. In the branching $\mathcal{F}_{\bar{x}}$, we get a 1 -clause containing only $y$. Furthermore, $\mathcal{F}_{\bar{x}}$ has at least two clauses containing $y$ because $y$ and $\bar{x}$ do not form a coincident pair in $\mathcal{F}$. We apply R-Rule 1 on $y$ and can further decrease the number of clauses by at least 2 . We get a branching vector at least $(3,5)$.

Case 3. $\mathcal{F}$ contains a $\left(2,3^{+}\right)$-literal (but no $\left(3,4^{+}\right)$or $\left(4^{+}, 3\right)$-literal): Now $\mathcal{F}$ contains only $\left(2,3^{+}\right)$-literals, $\left(3^{+}, 2\right)$-literals and ( 3,3 )-literals. We consider the following subcases.

Case 3.1. There is a 2 -clause $C=\{x, y\}$ containing a $\left(3^{+}, 2^{+}\right)$-literal $x$ : We do a deeper analysis by considering different cases.

Case 3.1.1. Each clause containing $\bar{x}$ is a 2 -clause: We branch on $x$. In the branching of $\mathcal{F}_{x}$, we will get at least two 1 -clauses. By applying R-Rule 1 on them, we can further reduce 2 clauses. In the branching of $\mathcal{F}_{\bar{x}}$, we get at least one 1-clause. By applying R-Rule 1 on it, we can further reduce 1 clause. So we can get a branching vector $(5,3)$ at least.

Next, we can assume that there is a literal $z \notin\{x, \bar{x}, y, \bar{y}\}$ appearing in a clause containing $\bar{x}$.

Case 3.1.2. At least one of $x$ and $y$ is a ( 3,3 -literal: We assume that $x$ is a ( 3,3 )-literal. We branch on $x$. In the branching of $\mathcal{F}_{\bar{x}}$, there is a 1 -clause $\{y\}$ and we can reduce at least one clause by applying R-Rule 1 on it. Because $z$ exists, by Lemma 8 we know that if only four clauses are removed in total, the remaining instance will be a good formula. In the branching of $\mathcal{F}_{x}$, three clauses are deleted and the remaining instance is also a good formula by Corollary 1 . We can branch with a branching vector $(3,4)$ with a good formula in each remaining branching or branch with a branching vector at least $(3,5)$.

Case 3.1.3. Both of $x$ and $y$ are $\left(3^{+}, 2\right)$-literals: We further consider two subcases.
If each clause containing $\bar{x}$ also contains $y$, then we branch on $y$. In the branching of $\mathcal{F}_{y}$, literal $x$ will become a $\left(2^{+}, 0\right)$-literal. We can reduce two more clauses by applying R-Rule 1 on $x$. In the branching of $\mathcal{F}_{\bar{y}}$, we will have a 1 -clause $\{x\}$. We can reduce at least one clause by applying R-Rule 1 on $\{x\}$. Then we can get a branching vector $(5,3)$ at least.

Otherwise, at most one clause containing $\bar{x}$ contains $y$. For this case, we branch on $x$. In the branching of $\mathcal{F}_{\bar{x}}$, we will have a 1 -clause $\{y\}$. We can reduce at least two clauses by applying R-Rule 1 on $\{y\}$. As $z$ exists, by Lemma 8 we know if just 4 clauses are removed in total, the remaining instance is a good formula. For the branching of $\mathcal{F}_{x}$, three clauses are deleted and we can apply Corollary 1. The remaining instance is also a good formula. So we get a branching vector $(3,4)$ with a good formula in each remaining branching or a branching vector covered by $(3,5)$.

Case 3.1.4. Literal $x$ is a $\left(3^{+}, 2\right)$-literal, $y$ is a $\left(2,3^{+}\right)$-literal, and no clause contains both of $y$ and $\bar{x}$ : We branch on $x$. In the branching of $\mathcal{F}_{x}$, literal $y$ will become a $\left(1^{-}, 0^{+}\right)$-literal. We can reduce at least one clause by applying R-Rule 1 or R-Rule 3 on $y$. In the branching of $\mathcal{F}_{\bar{x}}$, we will have a 1 -clause $\{y\}$ and can reduce at least two clauses by applying R-Rule 1 . Thus, we can get a branching vector of $(4,4)$.

Case 3.1.5. Literal $x$ is a $\left(3^{+}, 2\right)$-literal, $y$ is a $\left(2,3^{+}\right)$-literal, and a clause contains both of $y$ and $\bar{x}$ : We branch on $x$.

Assume that there is a 2 -clause other than $C$ containing $x$. In the branching of $\mathcal{F}_{x}$, we can further decrease the number of clauses by at least 1 by applying R-Rule 3 on $y$. In the branching
of $\mathcal{F}_{\bar{x}}$, we will get at least two 1 -clauses and can further decrease the number of clauses by at least 2 by applying R-Rule 1 . We can get a branching vector covered by $(4,4)$.

Otherwise, the other two clauses containing $x$, denoted by $C_{1}$ and $C_{2}$, are both $3^{+}$-clauses. We can simply assume that $C_{i}(i=1,2)$ does contains $C=(x, y)$ or both a literal and its negation, since for this case we can simply delete $C_{i}$ without branching. Thus $C_{1} \cup C_{2}$ will contain at least two different literals $z_{1}$ and $z_{2}$ that are also different from $x, \bar{x}, y$ and $\bar{y}$. If $x$ is a $\left(4^{+}, 2\right)$-literal, in the branching of $\mathcal{F}_{x}$, we reduce at least four clauses directly and leave a $\left(1,0^{+}\right)$literal $y$. By applying R-Rule 1 or R-Rule 3 on $y$, we can further reduce at least one clause. So we can reduce at least five clauses for this case. Next, we assume that $x$ is a (3,2)-literal. For this case, in $\mathcal{F}_{x}$, literal $y$ will become a $\left(1,1^{+}\right)$-literal, and literals $z_{1}$ and $z_{2}$ will become two different $\left(2^{-}, 0^{+}\right)$-literals (also different from $y$ and $\bar{y}$ ). By Lemma 9 we know that after resolving $y$ in $\mathcal{F}_{x}$, we can reduce one clause with the resulting formula being good or reduce at least two clauses directly. So in the branching of $\mathcal{F}_{x}$, we can either reduce four clauses leaving a good formula or reduce at least five clauses. In the other branching of $\mathcal{F}_{\bar{x}}$, we get a 1 -clause $\{y\}$, after applying R-Rule 1 on it we can further reduce one clause. If only three clauses are reduced in this branching, then the remaining formula is good. The reason is as below. In Case 3, $\mathcal{F}$ contains only $\left(2,3^{+}\right)$-literals, $\left(3^{+}, 2\right)$-literals and (3, 3$)$-literals. There is a literal $z \notin\{x, \bar{x}, y, \bar{y}\}$ appears in a clause containing $\bar{x}$ (after Case 3.1.1). For this case, $z$ will be a $\left(2^{-}, 0^{+}\right)$-literal or $\left(0^{+}, 2\right)$-literal in the remaining formula and then the remaining formula is good. We can branch with a branching vector $(4,3)$ leaving a good formula in each branching or a branching vector covered by $(5,3)$ or $(4,4)$.

Case 3.2. There is a 2 -clause $C=\{x, y\}$ containing two $\left(2,3^{+}\right)$-literals: We consider two subcases.

Case 3.2.1. There is no clause containing both of $y$ and $\bar{x}$ : We branch on $x$. In the branching of $\mathcal{F}_{x}$, literal $y$ will become a $\left(1^{-}, 2^{+}\right)$-literal. We can reduce one more clause by applying R-Rule 3 on $y$. In the branching of $\mathcal{F}_{\bar{x}}$, a 1 -clause $\{y\}$ is created and there are two clauses containing $y$. We can reduce two more clauses by applying R-Rule 1 on $y$. We get a branching vector of $(3,5)$.

Case 3.2.2. There is a clause $D$ containing both of $y$ and $\bar{x}$ : If $D$ is also a 2-clause, then there are two 2 -clauses $\{x, y\}$ and $\{\bar{x}, y\}$. We simply assign $y=1$ without branching. Next, we assume that $D$ is a $3^{+}$-clause.

If $D$ is a 3 -clause, we branch on $y$. In the branching of $\mathcal{F}_{y}$, literal $x$ will become a $\left(1^{-}, 2^{+}\right)$literal. We can reduce one more clause by applying R-Rule 3 on $x$. In the branching of $\mathcal{F}_{\bar{y}}$, we will get two -clauses $\{x\}$ and $\{z\}$, where $z$ is the third literal in $D$. By applying R-Rule 1 on $\{x\}$ and $\{z\}$, we can reduce two more clauses. We get a branching vector of $(3,5)$.

Else $D$ is a $4^{+}$-clause, and we branch on $x$. In the branching of $\mathcal{F}_{x}$, literal $y$ will become a $\left(1,2^{+}\right)$-literal. After applying R-Rule 3 on $y$, we reduce one more clause leaving a good formula, because $D$ contains at least two literals other than $y$ and $\bar{x}$ and then there is a coincident pair after applying R-Rule 3 on $y$. In the branching of $\mathcal{F}_{\bar{x}}$, we will get a 1 -clause $\{y\}$. We can reduce one more clause by applying R-Rule 1 on it. Same as before, if just 4 clauses are removed, the remaining instance is good. Thus, we can either get a branching vector $(3,4)$ with a good formula in each remaining branching or a branching vector covered by $(3,5)$.

Next, we assume that there is no 2-clause.
Case 3.3. There is a clause in $\mathcal{F}$ containing both a (3,3)-literal $x$ and a $\left(2,3^{+}\right)$-literal $y$ : Let $C_{1}, C_{2}$ and $C_{3}$ be the three clauses containing $x$, where we assume that $C_{1}$ also contains $y$. Let $C_{4}$ be the other clause containing $y$. We first branch on $x$ with a branching vector $(3,3)$. We may decrease the number of clauses more by applying reduction rules for different cases.

Case 3.3.1. $C_{4}=C_{2}$ or $C_{4}=C_{3}$ : This means $\{x, y\}$ is a coincident pair. In the branching $\mathcal{F}_{x}$, the literal $y$ becomes a $\left(0,2^{+}\right)$-literal. We can further remove at least two clauses by applying R-Rule 1 on $y$. We get a branching vector $(5,3)$. Next, we assume that $C_{4} \neq C_{2}$ or $C_{3}$.

Case 3.3.2. $C_{4} \neq C_{2}$ and $C_{4} \neq C_{3}$ : Notice that $C_{2}$ and $C_{3}$ are $3^{+}$-clauses and each of them will contain a literal different from $\{x, \bar{x}, y, \bar{y}\}$. In $\mathcal{F}_{x}$, there is a $\left(1,1^{+}\right)$-literal $y$ and two different $\left(2^{-}, 0^{+}\right)$-literals different from $\{x, \bar{x}, y, \bar{y}\}$. So it satisfies the condition in Lemma 9 . After resolving $y$ in $\mathcal{F}_{x}$, we can further either reduce one clause leaving a good formula or reduce at least two clauses. In the branching of $\mathcal{F}_{\bar{x}}$, we reduce three clauses directly and the remaining formula is good according to Corollary (1) So the branching vector is either $(4,3)$ with a good formula in each branching or a vector covered by $(5,3)$.

Lemma 10 For a reduced instance $\mathcal{F}$ without $\left(3^{+}, 4^{+}\right)$-literals, if there is no 2-clause and no clause contains both $a\left(2,3^{+}\right)$-literal and a $(3,3)$-literal, then either there is no $\left(2,3^{+}\right)$-literal or there is a clause containing at least three $\left(2,3^{+}\right)$-literals.

Proof. Since $\mathcal{F}$ is a reduced instance, we know that the degree of any literal is at least 2 and there is no $(2,2)$-literal. Note that there is also no $\left(3^{+}, 4^{+}\right)$-literal. Thus, the formula contains only $\left(2,3^{+}\right)$-literals, $\left(3^{+}, 2\right)$-literals and $(3,3)$-literals. We assume that there is at least one $\left(2,3^{+}\right)$-literal otherwise the lemma trivially holds. It is impossible that each clause containing a ( 2,3 )-literal also contains a ( $3^{+}, 2$ )-literal because this case would be reduced by R-Rule 4 . So there is a clause containing only $\left(2,3^{+}\right)$-literals. Since there is no 2 -clause. We know that the clause contains at least three $\left(2,3^{+}\right)$-literals.

By Lemma 10, we know that the remaining case is as follows.
Case 3.4. There is a $3^{+}$-clause $C$ containing at least three $\left(2,3^{+}\right)$-literals $\left\{x_{1}, x_{2}, x_{3}\right\}$ : Let $C_{i}$ be the other clause containing $x_{i}(i=1,2,3)$, where it is possible two of $C_{1}, C_{2}$ and $C_{3}$ are the same.

Case 3.4.1. Two literals in $\left\{x_{1}, x_{2}, x_{3}\right\}$, say $x_{1}$ and $x_{2}$, form a coincident pair: We branch on $x_{1}$ with a branching vector $(2,3)$ first. In the branching of $\mathcal{F}_{x_{1}}$, literal $x_{2}$ will become a $\left(0,3^{+}\right)$literal and we reduce three clauses by applying R-Rule 1 on $x_{2}$. So we can get a branching vector of $(5,3)$.

Case 3.4.2. At least one of $C_{1}, C_{2}$ and $C_{3}$ contains a negation of $x_{1}, x_{2}$ or $x_{3}$ : Without loss of generality we assume that $C_{2}$ contains a negation of $x_{1}$. We first branch on $x_{1}$ with a branching vector $(2,3)$. In the branching of $\mathcal{F}_{x_{1}}$, each of $x_{2}$ and $x_{3}$ will become a $\left(1,1^{+}\right)$-literal. We can further reduce the number of clauses by at least 2 by applying R-Rule 3 on $x_{2}$ and $x_{3}$ one by one. In the branching of $\mathcal{F}_{\bar{x}_{1}}$, after deleting the three clauses containing $\overline{x_{1}}$ (including $C_{2}$ ), the degree of $x_{2}$ is at most 1 . We can reduce one more clause by applying reduction rules on $x_{2}$. Thus, we can branch with a branching vector $(4,4)$.

Case 3.4.3. None of Case 3.4.1 and Case 3.4.2 happens: We first branch on $x_{1}$ with a branching vector $(2,3)$. In the branching of $\mathcal{F}_{x_{1}}$, each of $x_{2}$ and $x_{3}$ will become a $\left(1,3^{+}\right)$-literal. We can reduce two more clauses by applying R-Rule 3 on $x_{2}$ and $x_{3}$ one by one. Furthermore, the remaining instance is a good formula, because applying R-Rule 3 will create coincident pairs in this case. In the branching $\mathcal{F}_{\overline{x_{2}}}$, the formula is a good formula by Corollary 1 . We get a branching vector $(3,4)$ with a good formula in each branching.

Case 4. $\mathcal{F}$ contains only $(3,3)$-literals: Since $\mathcal{F}$ is a good formula, we know that there is either a coincident pair or a 2-clause.

Case 4.1. $\mathcal{F}$ contains a coincident pair $\{x, y\}$ : We branch on $x$ into two branchings $\mathcal{F}_{x}$ and $\mathcal{F}_{\bar{x}}$, and distinguish two subcases to analyze the branching operation.

Case 4.1.1. Three clauses contain $x$ and $y$ simultaneously: In the branching of $\mathcal{F}_{x}$, the literal $y$ will become a $(0,3)$-literal and we can further decrease the number of clauses by at least 3 by applying R-Rule 1 . So we can get a branching vector $(3,6)$ at least.

Case 4.1.2. Only two clauses contain $x$ and $y$ simultaneously: we assume without loss of generality that no pair of literals appear in more than two clauses simultaneously now.

Assume that one of the clauses containing $x$ is a 2-clause $\{x, w\}$, where $w$ can be $y$. In the branching of $\mathcal{F}_{x}$, we can apply R-Rule 3 on $y$ to further reduce 1 clause. In the branching of $\mathcal{F}_{\bar{x}}$, we can apply R-Rule 1 on $w$ to further reduce 1 clause. The branching vector will be covered by $(4,4)$.

Next, we assume that any of the three clauses containing $x$ also contains a literal other than $y$ and $\bar{y}$. At least two of the three literals are different because no pair of literals appear in three clauses as assumed. Let $z_{1}$ and $z_{2}$ be the two different literals. In $\mathcal{F}_{x}$, literal $y$ will become a $\left(1,1^{+}\right)$-literal and $z_{1}$ and $z_{2}$ will become $\left(2^{-}, 0^{+}\right)$-literals. The condition in Lemma 9 holds. After resolving $y$ in $\mathcal{F}_{x}$, we can further either reduce 1 clause leaving a good formula or reduce at least 2 clauses. In the branching of $\mathcal{F}_{\bar{x}}$, we reduce three clauses directly and the leaving formula is good according to Corollary The branching vector is either $(4,3)$ with a good formula in each branching or a vector covered by $(5,3)$.

Case 4.2. $\mathcal{F}$ does not contain a coincident pair but contains a 2 -clause $\{x, y\}$ : We branch on $x$ with a branching vector $(3,3)$. In the branching $\mathcal{F}_{\bar{x}}$, we will get a 1 -clause that only contains $y$. Furthermore, since $\mathcal{F}$ does not contain a coincident pair, we know that there are at least two clauses containing $y$ in $\mathcal{F}_{\bar{x}}$. We can apply R-Rule 1 on $y$ in $\mathcal{F}_{\bar{x}}$ to further reduce 2 clauses. Thus, we can get a branching vector covered by $(3,5)$.

## 7 Conclusion

SAT is one of the most widely studied NP-complete problems. There is a large number of references in the history, whether from the perspective of experimental algorithms or theoretical algorithms. Many fast solvers have been developed and they can solve medium-large sized instances within a reasonable running time bound. However, the theoretical research is relatively backward. It took us decades to improve the running time bound to $O^{*}\left(1.2226^{m}\right)$. According to the theoretical results, the size of the problems we can solve is much smaller than that of the problems solved by fast practical solvers. The gap between theoretical and experimental results is large. It is interesting to further explore the problem nature and reduce the gap, especially to accelerate the research of theoretical algorithms and explain the fast experimental algorithms.

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[^0]:    ${ }^{1}$ The notation $O^{*}$ suppresses all polynomially bounded factors. For two functions $f$ and $g$, we write $f(n)=$ $O^{*}(g(n))$ if $f(n)=g(n) n^{O(1)}$.

